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Supersymmetry and Solitons: $N = 2$ and $N = 0$

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This talk summarizes our recent work establishing an algebraic, model-independent basis for the existence of Bogomol'nyi bounds and Bogomol'nyi equations for topologically non-trivial solitons and instantons. Our arguments use supersymmetry in an essential way to understand both supersymmetric and non-supersymmetric theories. Our arguments are constructive and work in nearly any number of dimensions.

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1. Introduction and Overview

Theories with topologically non-trivial solitons and instantons¹ regularly exhibit Bogomol'nyi or self-duality bounds (the soliton energy or instanton action is bounded from below by the charge or instanton number, respectively), with field configurations that saturate such bounds satisfying first-order differential equations, called Bogomol'nyi or self-duality equations.² (We will refer to such bounds and equations collectively as *Bogomol'nyi relationships*.) Despite this regularity, these features have so far been understood only on a case-by-case basis, by invoking the equations of motion.

Similarly, in models with $N = 1$ supersymmetry and a conserved topological charge, one regularly finds a larger algebraic symmetry structure, namely $N = 2$ supersymmetry with the topological charge as a central charge,³ and yet this, too, has heretofore been understood only on a case-by-case basis.

We address these theoretical shortcomings here by establishing the above results for solitons and instantons[†] in a general, model-independent way. Our method is to obtain these results first in the supersymmetric case, and then to show that this implies the corresponding results in the generic non-supersymmetric case. Our use of supersymmetry to understand non-supersymmetric theories is technically very similar to the use of complex analysis to address question about real functions.

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‡ We will use the terms *soliton* and *instanton* to refer to any field configuration with non-trivial topological charge or instanton number. This liberal usage will be useful here.

Because of space limitations, we will be very brief here. The reader interested in the subtleties and technicalities should refer to our papers for a detailed accounting.⁴

2. Why Topological Charges Imply Extended Supersymmetry

We demonstrate here that a theory with an $N = 1$ supersymmetry and a conserved topological charge necessarily has an $N = 2$ supersymmetry in which the topological charge appears as the central charge. We first give the argument in $2 + 1$ dimensions, then generalize this to arbitrary higher dimension, and finally discuss the significance of the breakdown of our argument in $1 + 1$ dimensions.

A theory with $N = 1$ supersymmetry and a topologically conserved charge in $2 + 1$ dimensions has a conserved real spinor charge Q^α , with

$$\{Q^\alpha, Q^\beta\} = P^{\alpha\beta} \quad , \quad (1)$$

and a current J_μ for which $\partial^\mu J_\mu = 0$ can be derived without using the equations of motion, which means that one can write J_μ in terms of a vector potential A_ν via⁵

$$J_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \quad . \quad (2)$$

The gauge equivalent vector potentials A_μ and $A_\mu + \partial_\mu \chi$ produce the same topological current J_μ . Among the gauge equivalent potentials, there is one that is divergenceless; hereon, we use A_μ to refer to this particular potential. (Note that the theory has no gauge *symmetry*, so we are not gauge-fixing the theory.) Define

$$\tilde{S}_\mu^\alpha = [Q^\alpha, A_\mu] \quad . \quad (3)$$

Since supersymmetry transformations commute with translations, we have $\partial^\mu \tilde{S}_\mu^\alpha = 0$, i.e., \tilde{S}_μ^α is a conserved vector-spinor.

Under the original supersymmetry, \tilde{S}_μ^α transforms into the non-trivial conserved topological charge, so this conserved spinor current is neither trivial nor is it the original supercurrent. Consequently, \tilde{S}_μ^α must be a second conserved spinor current. Since under the original supersymmetry this new supercurrent transforms into the topological current, the theory is invariant under an $N = 2$ superalgebra with a central charge given by the topological charge.

Had we started from a different but gauge-equivalent potential, the corresponding vector-spinor produced would differ from the second supercurrent by an element of the kernel of the original supercharge. Modding out by this kernel gives another means of identifying the physical supercurrent from among these vector-spinors.

In higher dimensions, one can write a topologically conserved current as the curl of a divergenceless $d - 2$ -index antisymmetric tensor (where d is the spacetime dimension). The second supercurrent is then simply

$$\tilde{S}_{\mu_1}^\alpha = [Q^\alpha, A_{\mu_1\mu_2\cdots\mu_{d-2}}]\gamma^{\mu_2}\cdots\gamma^{\mu_{d-2}} \quad . \quad (4)$$

The rest of the argument proceeds as before.

Note that our construction explicitly breaks down in $1 + 1$ dimensions. This is as it must be. In $1 + 1$ dimensions, there are supersymmetric models in which the topological charge does not serve as a central charge. That our method handles arbitrary dimensions, but explicitly breaks down for $1 + 1$ dimensions, indicates that we have indeed identified a fundamental approach to this phenomenon.

3. Bogomol'nyi Relationships for Supersymmetric Soliton Theories

The extended superalgebra derived above implies that Bogomol'nyi relationships arise in any supersymmetric theory with a topological charge. This result is easy to derive, using a standard observation. For simplicity, we give only the $2+1$ dimensional case here.

From the above results, we see we have the algebraic relation

$$\{Q_L^\alpha, Q_M^\beta\} = P^{\alpha\beta}\delta_{LM} + T\epsilon^{\alpha\beta}\epsilon_{LM} \quad , \quad (5)$$

where Q_L^α , $L = 1, 2$, are the two supercharges, $P^{\alpha\beta}$ is the momentum, and T is the topological charge. Since this anticommutator is hermitian, its square is positive semi-definite; taking the trace of this square we obtain the Bogomol'nyi bound

$$M^2 - T^2 \geq 0 \quad , \quad (6)$$

where M is the rest mass. Furthermore, this bound is saturated only when the field configuration is annihilated by one of the supercharges, a condition represented by a set of first-order equations (since the supercharges can be represented by first-order differential operators). These, then, are the Bogomol'nyi equations of the theory.

4. Bogomol'nyi Relationships for Solitons in General

Consider a generic, non-supersymmetric Lagrangian \mathcal{L} which is a functional of some field(s) ϕ , and which has a conserved topological charge $T[\phi]$. The energy of a field configuration is given by a functional $E[\phi]$. We now demonstrate that this theory exhibits Bogomol'nyi relationships.

Now consider a supersymmetric extension of this theory. Such an extension has a Lagrangian \mathcal{L}_s which is a functional of the original field(s) ϕ and some additional field(s) ψ . (This notation is suggestive of scalar and fermionic fields, but in fact any theory can be suitably extended.⁴) This theory has a topological charge $T_s[\phi, \psi]$ and an energy functional $E_s[\phi, \psi]$. There are three important features of this extension:

1. The field configurations of the original theory are also field configurations of the extended theory.
2. Since the topological charge is conserved without reference to the equations of motion, the extended theory has the *exact same* topological charge as the original theory, and so $T_s[\phi, \psi] = T_0[\phi]$, irrespective of the value of ψ .
3. We can and do choose the extension such that $E_s[\phi, \psi = 0] = E_0[\phi]$.

The extended theory is a supersymmetric theory with a conserved topological charge. Thus, $E_s[\phi, \psi] \geq |T_s[\phi, \psi]|$ for any field configuration, and any field configuration that saturates this inequality satisfies first-order Bogomol'nyi equations. For field configurations for which $\psi = 0$, using points 2 and 3 above, the preceding inequality yields

$$E_0[\phi] \geq |T_0[\phi]| \quad . \tag{7}$$

In this way, we have just obtained the Bogomol'nyi bound of the original theory! By points 2 and 3 again, a field configuration that saturates the Bogomol'nyi bound of the original theory also saturates the supersymmetric Bogomol'nyi bound, and so this field configuration satisfies first-order differential equations when viewed as a configuration of the supersymmetric extension. These equations involve only ϕ (since $\psi = 0$), and so they are the sought-after Bogomol'nyi equations of the original theory, derived in a general way.

5. Instantons and Self-Duality in General

To extend these results to topologically non-trivial instantons is the obvious next step. Since instanton number is *not* a conserved charge, it cannot be woven directly into a superalgebra. Fortunately, we can still build on our previous results.

Consider a Euclidean field theory in d dimensions with action $S_d[\phi]$ which has a topological instanton number given by the functional $I_d[\phi]$. One can construct an associated $d + 1$ -dimensional Minkowskian theory, by making all fields functions of an additional time coordinate, and adding the necessary field components (e.g., temporal components to vector fields) and time derivatives to ensure Lorentz invariance. The Minkowskian theory has energy functional E_{d+1} . Furthermore, the instanton number of the Euclidean theory becomes a topologically conserved charge T_{d+1} in the Minkowskian theory, since the d -dimensional instanton number cannot change as the fields evolve continuously from one time-slice to the next in $d + 1$ dimensions.

The $d + 1$ dimensional theory exhibits Bogomol'nyi relationships, as we have argued above. But the instantons of the d -dimensional theory are static solitons of the Minkowskian theory, which thus obey $E_{d+1} \geq |T_{d+1}|$. When these field configurations are viewed as instantons, this inequality becomes $S_d \geq |I_d|$. Furthermore, if an instanton saturates this bound, it satisfies a Bogomol'nyi equation in $d + 1$ dimensions when viewed as a soliton. Because the field is static, this in fact is a first-order equation in d dimensions, that is, the Euclidean space instanton self-duality equation, now obtained in a perfectly general way, and in an essentially unified treatment with solitons.

6. Closing Remarks

Our methods have demonstrated in a model-independent way that Bogomol'nyi relationships must arise for topological solitons and instantons. It is the hidden hand of supersymmetry that makes such a general approach feasible. Our method offers other insights, as well. Our approach shows very naturally why zero modes in self-dual or Bogomol'nyi-saturating backgrounds must be associated with index theorems. We are also able to generalize to any model a result of D'Adda and Di Vecchia's⁷ that in certain theories the non-zero modes in a Bogomol'nyi-saturating soliton or instanton background are bose-fermi degenerate. Further, there are suggestive connections between our results and the construction of topological field theories from $N = 2$ supersymmetric ones.

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References

1. A.M. Polyakov, *JETP Lett.* **20** (1974) 194;
A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu.S. Tyupkin, *Phys. Lett.* **59B** (1975) 85;
G. 't Hooft, *Phys. Rev.* **D14** (1976) 3432.

2. M.K. Prasad and C.H. Sommerfield, *Phys. Rev. Lett.* **35** (1975) 760;
 E.B. Bogomol'nyi, *Sov. J. Nucl. Phys.* **24** (1976) 449;
 G. 't Hooft, *Phys. Rev. Lett.* **37** (1976) 8;
 A.M. Polyakov, *Nucl. Phys.* **B121** (1977) 429.
3. D. Olive and E. Witten, *Phys. Lett.* **78B** (1978) 97;
 J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo, and P.K. Townsend, *Phys. Rev. Lett.* **63** (1989) 2443;
 C. Lee, K. Lee, and E. Weinberg, *Phys. Lett.* **243B** (1990) 105;
 M. Cvetič, F. Quevedo, and S.-J. Rey, *Phys. Rev. Lett.* **67** (1991) 1836;
 J. Edelstein, C. Nunez, and F. Schaposnik, *Phys.Lett.* **B329** (1994) 39.
4. Z. Hlousek and D. Spector, *Nucl. Phys.* **B370** (1992) 143;
 Z. Hlousek and D. Spector, *Phys. Lett.* **283B** (1992) 75;
 Z. Hlousek and D. Spector, *Mod. Phys. Lett.* **A7** (1992) 3403;
 Z. Hlousek and D. Spector, *Nucl. Phys.* **B397** (1993) 173;
 Z. Hlousek and D. Spector, *Solitons and Instantons without Supersymmetry*, submitted for publication.
5. C. Cronström and J. Mickelsson, *J. Math. Phys.* **24** (1983) 2528;
 R. Jackiw in S. Treiman, R. Jackiw, B. Zumino, and E. Witten *Current Algebra and Anomalies* (World Scientific, Singapore, 1985).
6. R. Haag, J. Łopuszański, and M. Sohnius, *Nucl. Phys.* **B288** (1975) 257.
7. A. D'Adda and P. Di Vecchia, *Phys.Lett.* **73B** (1978) 162.